

Quantum Hamiltonian Eigenstates for a Free Transverse Field

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Abstract

We demonstrate that the quantum Hamiltonian operator for a free transverse field within the framework of the second quantization reveals an alternative set of states satisfying the eigenstate functional equations. The construction is based upon extensions of the quadratic form of the transverse Laplace operator which are used as a source of spherical basis functions with singularity at the origin. This basis then naturally takes place of the one of plane or spherical waves in the process of Fourier or spherical variable separation.

Introduction

The second quantization approach [1], [2] has been the framework for constructing the quantum field theory since the time of its inception in the first half of the 20th century. Later in the development, for the purpose of practical calculations of the scattering matrix elements, a wide recognition was given to the technique of Feynman's diagrams, which is based on the Lagrangian formulation of the classical theory. Unlike this latter technique, one of the advantages of the second (canonical) quantization is that it provides a description in terms of the quantum Hamiltonian operator. In a correctly defined quantum system the latter object must be a self-adjoint operator in Hilbert space. The finite-dimensional examples [3], [4] show that upon renormalization and the removal of singularities the Hamiltonian nominee may well be a symmetric but still not a self-adjoint operator representing a free particle on a restricted space of states. Such a candidate can be extended to a self-adjoint operator, however this procedure is ambiguous as it requires introduction of an extension parameter (the dimensional transmutation phenomenon as of [4], [6]). A similar effect is seemingly observed in the systems

of infinite number of harmonic oscillators. We shall argue that the quadratic part of the quantum Hamiltonian of a free transverse vector field

$$\mathcal{H}_0 = \int_{\mathbb{R}^3} \left(-\frac{\delta}{\delta A_k(\vec{x})} \frac{\delta}{\delta A_j(\vec{x})} + \Delta A_j(\vec{x}) A_j(\vec{x}) \right) d^3x, \quad \partial_k A_k = 0,$$

which appears, for example, in electrodynamics or as a result of renormalization of a gauge theory, is a limiting case of self-adjoint extension of some symmetric operator defined on a restricted set of states. At the same time generic self-adjoint extensions turn out to be dependent on an extension parameter and for that reason do not possess scale invariance.

Due to the lack of adequate definition of a scalar product on the space of functionals, which describe states of the stationary picture of the quantum field theory, we shall not make strict statements about the self-adjointness or symmetricity. Instead, we shall provide a sketch for the new vacuum state and its excitations (the Fock space). These states satisfy the equations for “eigenstate” functionals and form a hierarchy of creation and annihilation of particles. It is natural to demand that these equations match the functional equations

$$\mathcal{H}_0 \Phi_{\sigma_n}(A) = \Lambda_{\sigma_n} \Phi_{\sigma_n}(A),$$

for eigenstates of Hamiltonian \mathcal{H}_0 , but at the same time they could be defined on a set of functionals which satisfy different conditions in the vicinity of the “boundary” functions. For the role of such boundary points in the configuration space, where the boundary conditions of the new functional space are set, one can take the functions with singularities behaving as

$$\vec{A}(x) \sim \frac{\vec{A}_0}{|x|}, \quad |x| \rightarrow 0, \quad \vec{x} \in \mathbb{R}^3. \quad (1)$$

The possible self-adjoint extensions of the theory, therefore, will depend on a certain preferred point in the three-dimensional space, which should be associated with a localization in the interaction terms. However as self-interaction is “turned on”, such extensions of the Hamiltonian and the related states will most likely turn out to be unstable. They, however, may still contribute to the scattering matrix as intermediate states for particles interacting via the transverse field.

For the sake of brevity we shall use the following notations for the scalar and vector products

$$\vec{A} \cdot \vec{B} = A_j B_j, \quad (\vec{A} \times \vec{B})_n = \epsilon_{njk} A_j B_k, \quad j, k, n = 1, 2, 3,$$

and we shall always assume summation in the repeated indices.

1 Finite-dimensional examples with singular interactions

We start with the finite-dimensional examples from quantum mechanics with the intent to generalize some of their properties to the infinite-dimensional case. Let

$$H_\varepsilon = \Delta + \varepsilon \delta(x) = -\frac{\partial^2}{\partial x_k^2} + \varepsilon \delta(x)$$

be the Hamiltonian of a particle existing in the two- or three-dimensional space and interacting with a δ -potential centered at the origin. Hamiltonian H_ε does not have a correct definition in terms of a closed operator in Hilbert space. One can, however, consider the action of H_ε on the set of smooth functions falling off towards the origin along with their derivatives. This action corresponds to a symmetric operator

$$H : Hf(\vec{x}) = \Delta f(\vec{x}) = -\frac{\partial^2}{\partial x_k^2} f(\vec{x}),$$

which, evidently, does not account for potential $\varepsilon \delta(\vec{x})$. In terms of the spherical coordinates in two-dimensional

$$\vec{x} = \vec{x}(r, \varphi) = \begin{pmatrix} r \cos \varphi \\ r \sin \varphi \end{pmatrix}, \quad \begin{array}{l} 0 \leq r, \\ 0 \leq \varphi < 2\pi \end{array}$$

or three-dimensional space

$$\vec{x} = \vec{x}(r, \psi, \varphi) = \begin{pmatrix} r \cos \psi \cos \varphi \\ r \cos \psi \sin \varphi \\ r \sin \psi \end{pmatrix}, \quad \begin{array}{l} 0 \leq r, \\ 0 \leq \psi \leq \pi, \\ 0 \leq \varphi < 2\pi \end{array} \quad (2)$$

the action of operator H_0 has the following form. If a scalar function $f(\vec{x})$ is represented in terms of a sum of spherical harmonics $e^{il\varphi}$ or $Y_{lm}(\psi, \varphi)$ with coefficients depending on the radial variable,

$$\begin{aligned} f_2(\vec{x}) &= f_2(\vec{x}(r, \varphi)) = \sum_{0 \leq l} \frac{1}{\sqrt{r}} u_l(r) \frac{e^{il\varphi}}{\sqrt{2\pi}}, \\ f_3(\vec{x}) &= f_3(\vec{x}(r, \psi, \varphi)) = \sum_{0 \leq |m| \leq l} \frac{1}{r} u_{lm}(r) Y_{lm}(\psi, \varphi), \end{aligned}$$

then the corresponding operation Δ acts as follows

$$\begin{aligned}\Delta f_2(\vec{x}) &= \sum_{0 \leq l} \frac{1}{\sqrt{r}} T_{l-\frac{1}{2}} u_l(r) \frac{e^{il\varphi}}{\sqrt{2\pi}}, \\ \Delta f_3(\vec{x}) &= \sum_{0 \leq |m| \leq l} \frac{1}{r} T_l u_{lm}(r) Y_{lm}(\psi, \varphi),\end{aligned}$$

where

$$T_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}, \quad (3)$$

$$T_l^{-1}(r, s) = \frac{1}{2l+1} \left(\frac{s^{l+1}}{r^l} \theta(r-s) + \frac{r^{l+1}}{s^l} \theta(s-r) \right). \quad (4)$$

Note that due to the orthonormality of the sets of spherical harmonics, each of the scalar products

$$(f, g)_{\mathbb{R}^2} = \int_{\mathbb{R}^2} \overline{f(\vec{x})} g(\vec{x}) d^2x, \quad (f, g)_{\mathbb{R}^3} = \int_{\mathbb{R}^3} \overline{f(\vec{x})} g(\vec{x}) d^3x,$$

transfers to the coefficient functions $u(r)$ as a plain scalar product on the half-axis

$$(u, v) = \int_0^\infty \overline{u(r)} v(r) dr. \quad (5)$$

Operators $T_{l-\frac{1}{2}}$ and T_l , defined on the set of smooth functions vanishing at the origin along with their derivatives, are essentially self-adjoint with respect to scalar product (5) at $l \geq 1$. At the same time, operators $T_{-\frac{1}{2}}$, T_0 that act on the latter set are symmetric operators with deficiency indices $(1, 1)$. Their self-adjoint extensions $T_{-\frac{1}{2}}^\kappa$, T_0^κ have continuous spectrum eigenfunctions that look like

$$\begin{aligned}v_\lambda(r) &= \sqrt{\lambda r} (\alpha_{v\lambda} J_0(\lambda r) + \beta_{v\lambda} Y_0(\lambda r)), \quad T_{-\frac{1}{2}}^\kappa v_\lambda = \lambda^2 v_\lambda \\ u_\lambda(r) &= \alpha_{u\lambda} \sin \lambda r + \beta_{u\lambda} \cos \lambda r, \quad T_0^\kappa u_\lambda = \lambda^2 u_\lambda \\ \alpha_{\{u,v\}\lambda} &= \alpha_{\{u,v\}}(\lambda, \kappa), \quad \beta_{\{u,v\}\lambda} = \beta_{\{u,v\}}(\lambda, \kappa),\end{aligned}$$

along with, possibly, some eigenfunctions of the discrete spectrum. The actions of extensions $T_{-\frac{1}{2}}^\kappa$, T_0^κ match the differential operations $T_{-\frac{1}{2}}$ and T_0 , correspondingly.

Returning to Cartesian coordinates, therefore, symmetric operators H can be extended to self-adjoint operators H_2^κ, H_3^κ defined on the set of functions satisfying the asymptotic conditions

$$\lim_{r \rightarrow 0} \frac{f(\vec{x}(r))}{\ln r} = \kappa \lim_{r \rightarrow 0} (f(\vec{x}(r)) - \lim_{r' \rightarrow 0} \frac{f(\vec{x}(r'))}{\ln r'} \ln r), \quad (6)$$

or

$$\lim_{r \rightarrow 0} r f(\vec{x}(r)) = -\kappa \lim_{r \rightarrow 0} (1 + r \frac{\partial}{\partial r}) f(\vec{x}(r)), \quad (7)$$

at the origin (see Eqs. (3.43), (3.44) in [4]). Action of H_2^κ, H_3^κ , still matches the sum of squares of the second derivatives Δ on the corresponding set of functions on \mathbb{R}^2 or \mathbb{R}^3 .

Extensions H_2^κ, H_3^κ depend on parameter κ , the dimension of which originates from the presence of dimensionality of operator H : $[H] = [x]^{-2}$. From physical perspective one can say that H_2^κ, H_3^κ appear as a result of renormalization of the respective operators H_ε at $\varepsilon \rightarrow 0$. Meanwhile the singular functions with asymptotics (6), (7) at the origin, emerging in the domains of H_2^κ, H_3^κ , represent the remnant of the renormalized singular interaction $\varepsilon \delta(\vec{x})$. In the case of a particle in two-dimensional space one has the phenomenon of dimensional transmutation — a dimensionless parameter ε is replaced with a dimensional parameter κ during renormalization [5], [6].

As another example one can consider two- and three-dimensional operators of the type

$$\Delta + \frac{\varepsilon}{|x|^2} = -\frac{\partial^2}{\partial x_k^2} + \frac{\varepsilon}{|x|^2}, \quad (8)$$

with ε being a dimensionless parameter. Such operators are closed symmetric operators at finite ε in some vicinity of zero (in two-dimensional case ε has to be positive). When drawing a function satisfying the eigenstate equation one observes that the increase of the divergence by $|x|^{-2}$ originating from the action of the potential can cancel the divergence from the action of the Laplacian. Therefore, operator (8) has an alternative basis of locally square-integrable “eigenfunctions” behaving as $|x|^\eta$ near the origin ($\eta = -\sqrt{\varepsilon}$ in two dimensions and $\eta = -\frac{1}{2}(1 + \sqrt{1 + 4\varepsilon})$ in three dimensions), that is, it allows self-adjoint extensions (this by no means is a mathematically strict explanation of the Frobenius method [7]). One can show that in the limit $\varepsilon \rightarrow 0$ these extensions continuously turn into the corresponding operators H_2^κ and H_3^κ .

2 Three-dimensional transverse field theory

From the perspective of theory of operators in Hilbert space, the example of the last section shows that the restriction of the domain of Laplacian Δ to the set of smooth functions vanishing at the origin leads to a symmetric operator and an ambiguity in the definition of the Hamiltonian of the system. At the same time, the interaction, which disappears during renormalization, only serves as a catalyst for that ambiguity by selecting a preferred point in the space. In this work we shall try to generalize the experience of the finite-dimensional example to the case of field theory. We cannot really speak of self-adjointness in the case of an infinite dimensional configurational space as long as we do not have a possibility to define the scalar product on a wide enough class of functionals. For this reason we shall limit ourselves to the description of eigenvectors of such systems after renormalization, and shall provide an instructive example of an alternative set of vacuum state and its excitations.

Consider the following Hamiltonian functional of the classical mechanics

$$\begin{aligned} \mathcal{H}_\varepsilon = \iint_{\mathbb{R}^3} E_j(\vec{x}) P_{kj}^\varepsilon(\vec{x}', \vec{x}) P_{kj'}^\varepsilon(\vec{x}', \vec{y}) E_{j'}(\vec{y}) d^3x d^3x' d^3y + \\ + \int_{\mathbb{R}^3} ((\partial_k A_j(\vec{x}))^2 + \varepsilon(A^3(\vec{x}) + \dots)) d^3x. \end{aligned} \quad (9)$$

where $A_k^a(x)$, $E_k^a(x)$ are the fields of generalized coordinates and their conjugate momenta in the three-dimensional space, which satisfy the transversality conditions

$$\partial_k A_k^a = 0, \quad \partial_k E_k^a = 0. \quad (10)$$

We have denoted as $\varepsilon(A^3 + \dots)$ the homogeneous terms of dimension $[x]^{-4}$ of higher order in coordinates A_k^a , which also include the interaction. Matrix P_{kj}^ε is the projector from the transverse to the covariant-transverse field sets,

$$P_{kj}^\varepsilon = \delta_{kj} - \partial_k M^{-1}(\partial_j - \varepsilon A_j), \quad M = (\partial_j - \varepsilon A_j) \partial_j,$$

and ε is a small dimensionless parameter of the theory. Fields $A_k^a(x)$, $E_k^a(x)$ can also have an internal symmetry index a which is everywhere assumed to be summed upon. The action of the covariant derivative (and of all objects that contain it) may be non-trivial in this index

$$(\partial_k - \varepsilon A_k)^{ab} B^b = \partial_k B^a - \varepsilon A_k^c t^{abc} B^b.$$

We shall be only considering the quadratic terms, for which the non-triviality in this index reduces to mere summation, provided that the matrices t^{abc} are orthogonal for different c . This way the components corresponding to different values of the upper index of field $A_k^a(x)$ will separate.

An actual physical example of the Hamiltonian of type (9) is given in the third chapter of book [8]. Indeed, in Eq. (2.5) therein the following Hamiltonian density is presented,

$$h = \frac{1}{2}(E_k^a)^2 + \frac{1}{4}(\partial_k A_j - \partial_j A_k - \varepsilon[A_j, A_k]^a)^2, \quad (11)$$

where $\vec{A}(\vec{x})$ is the transverse field, and the conjugate momentum E_k^a is placed a constraint (2.41) upon,

$$(\partial_k - \varepsilon A_k)^{ab} E_k^b = 0. \quad (12)$$

After splitting momentum $\vec{E}(\vec{x})$ into its longitudinal and transverse components

$$E_k = E_k^L + E_k^T, \quad \partial_k E_k^T = 0, \quad E_k^L = \partial_k \xi(\vec{x})$$

we obtain from condition (12) that

$$\begin{aligned} \xi(\vec{x}) &= -M^{-1}(\partial_l - \varepsilon A_l)E_l^T, \quad M = (\partial_j - \varepsilon A_j)\partial_j, \\ E_k &= (\delta_{kl} - \partial_k M^{-1}(\partial_l - \varepsilon A_l))E_l^T, \end{aligned}$$

and the Hamiltonian density (11) after integrating by parts is transformed to the following form of Eq. (9)

$$\begin{aligned} h &= \frac{1}{2}((\delta_{kl} - \partial_k M^{-1}(\partial_l - \varepsilon A_l))E_l^T)^2 + \frac{1}{2}(\partial_k A_j^a)^2 + \\ &+ \varepsilon \partial_k A_j^a [A_j, A_k]^a + \frac{1}{2}\varepsilon^2([A_j, A_k]^a)^2. \end{aligned}$$

The quantum counterpart \mathcal{H}_ε of Hamiltonian \mathcal{H}_ε in the coordinate representation acts on functionals $\Phi(A_j(\vec{x}))$ in accord with expression (9) with $A_j(\vec{x})$ changed to the operator of multiplication by $A_j(\vec{x})$ and $E_j(\vec{x})$ changed to variation $\frac{\delta}{i\delta A_j(\vec{x})}$. (We assume that the Planck constant equals one and do not discuss the ordering of the canonical pairs.)

$$\begin{aligned} \mathcal{H}_\varepsilon &= - \iint_{\mathbb{R}^3} P_{kj}^\varepsilon(\vec{x}', \vec{x}) P_{kj'}^\varepsilon(\vec{x}', \vec{y}) \frac{\delta}{\delta A_j(\vec{x})} \frac{\delta}{\delta A_{j'}(\vec{y})} d^3x d^3x' d^3y + \\ &+ \int_{\mathbb{R}^3} ((\partial_k A_j(\vec{x}))^2 + \varepsilon(A^3(\vec{x}) + \dots)) d^3x. \end{aligned}$$

During the renormalization procedure when $\varepsilon \rightarrow 0$ the higher order terms $\varepsilon(A^3 + \dots)$ disappear, while projector P_{kj}^ε turns into the orthogonal projector onto the transverse component

$$P_{kj}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} P_{kj} = \delta_{kj} - \partial_k \partial^{-2} \partial_j, \quad P_{kj}^T = P_{kj}, \quad P_{kn} P_{nj} = P_{kj}. \quad (13)$$

However, in general there is a difference between the result \mathcal{H}_{ren} of renormalization of \mathcal{H}_ε at $\varepsilon \rightarrow 0$ and the first term \mathcal{H}_0 in the expansion of \mathcal{H}_ε in ε in the vicinity of zero

$$\mathcal{H}_\varepsilon = \mathcal{H}_0 + \sum_{n \geq 1} \frac{\partial^n \mathcal{H}_\varepsilon}{\partial \varepsilon^n} \Big|_{\varepsilon=0} \frac{\varepsilon^n}{n!}. \quad (14)$$

Treatment of divergences in \mathcal{H}_ε requires introduction of regularization parameters, which in turn are related to ε (see the resolvent example in [5] or [6]). As a result some or all terms in expansion (14) may become finite even at $\varepsilon = 0$ and the renormalized Hamiltonian \mathcal{H}_{ren} will not be equal \mathcal{H}_0 . Below we propose that the contribution of these finite terms is such that it only forces \mathcal{H}_{ren} to acquire an alternative set of eigenstates, while preserving the action of \mathcal{H}_0 . We have already seen how this picture is realized in the finite-dimensional examples in the last section.

More specifically, the present proposal relates to the fact that Hamiltonian \mathcal{H}_ε might have singularities via projector P_{kj}^ε around the boundary functions, which locally behave as $|x|^{-1}$. Higher order homogeneous and interaction terms have singularities of the same kind. In analogy to example (8), these two types of singularities can cancel each other, and in this way supply the renormalized quantum Hamiltonian with a domain having new boundary conditions and, correspondingly, different spectral properties.

In order to see this better, let us consider the action of the quantum Hamiltonian operator \mathcal{H}_0 from (14) upon functionals $\Phi(A)$,

$$\mathcal{H}_0 \Phi(A) = - \iint_{\mathbb{R}^3} \frac{\delta}{\delta A_k(\vec{x})} P_{kj}(\vec{x}, \vec{y}) \frac{\delta}{\delta A_j(\vec{y})} d^3 x d^3 y \Phi(A) + Q(A) \Phi(A). \quad (15)$$

Here P_{kj} is the projector (13) onto the transverse subspace, and $Q(A)$ is the quadratic form of the Laplace operator Δ ,

$$Q(A) = \int_{\mathbb{R}^3} (\partial_k A_j(\vec{x}))^2 d^3 x = \quad (16)$$

$$= - \int_{\mathbb{R}^3} A_j(\vec{x}) \frac{\partial^2}{\partial x_k^2} A_j(\vec{x}) d^3 x = \int_{\mathbb{R}^3} A_j(\vec{x}) \Delta A_j(\vec{x}) d^3 x. \quad (17)$$

The unnormalized vacuum state of operator \mathcal{H}_0 can be constructed as a Gaussian functional

$$\Phi_0(A) = \exp\left\{-\frac{1}{2}(A, P\Delta^{\frac{1}{2}}PA)\right\}. \quad (18)$$

And then the n -particle excitations (the Fock states) are

$$\Phi_{\sigma_n}(A) = \iint \sigma_n^{j_1 \dots j_n}(\vec{x}_1, \dots \vec{x}_n) b_{j_1}(\vec{x}_1) \dots b_{j_n}(\vec{x}_n) d^3x_1 \dots d^3x_n \Phi_0(A),$$

where σ_n are some Bose-Einstein symmetric functions, and $b_j(\vec{x})$ are the creation operators from the corresponding pairs

$$b_j = P_{jk} \left(\frac{\delta}{\delta A_k} - \Delta_{kj'}^{\frac{1}{2}} A_{j'} \right), \quad a_j = P_{jk} \left(\frac{\delta}{\delta A_k} + \Delta_{kj'}^{\frac{1}{2}} A_{j'} \right),$$

of creation and annihilation operators. Here quite essential is the fact that projector P commutes with Δ , and hence, with an arbitrary function thereof — for example with $\Delta^{\frac{1}{2}}$ (Δ should be defined as a self-adjoint operator).

Quantum operator \mathcal{H}_0 intermixes functionals Φ_{σ_n} , however, as is easy to see, it leaves n -particle subspaces invariant. For further diagonalization it is necessary to pass to the spectral representation of operator Δ , which we shall do below in the framework of a more general approach. The main idea of that approach is to construct an alternative vacuum state via a method which, in analogy to the second quantization, can be called the method of second self-adjoint extensions.

2.1 Method of second self-adjoint extensions

In the case when the quantum Hamiltonian has the form of Eq. (15), and the positive closed quadratic form $Q(A)$ admits non-trivial extensions, there arises a natural way of constructing an alternative set of “eigenstates” of operator \mathcal{H}_0 .

In general a closed semi-bounded quadratic form $Q(A)$ can be defined by means of a closed operator S , symmetric or self-adjoint in scalar product (\cdot, \cdot) , via a natural formula

$$Q(A) = (A, SA) = (SA, A).$$

Herein domain \mathcal{D}_S of operator S is contained in domain \mathcal{D}_Q of form Q , and the latter, generally, differs from the former quite significantly. (One can

see, that for the field A in (17) to be in the domain of Δ it should be twice differentiable, while the integral in (16) requires the existence only of the first derivative of A .) As long as $Q(A)$ is semi-bounded and S is symmetric, it allows self-adjoint extensions S_κ one of which – Friedrichs extension [9] – also defines form Q , while the rest of the extensions, provided they are semi-bounded, define different quadratic forms Q_κ (for general material on quadratic forms see section VIII.6 of book [11]). These quadratic forms in certain cases (in fact, in a large number of simple examples) are extensions of the original form

$$Q \subset Q_\kappa,$$

that is, the domain of Q is contained in the closure of the domain of Q_κ

$$\mathcal{D}_Q \subset \overline{\mathcal{D}_{Q_\kappa}},$$

and for all vectors A from \mathcal{D}_Q the equality

$$Q(A) = Q_\kappa(A), \quad A \in \mathcal{D}_Q,$$

is obeyed. In particular, [12] gives spherically symmetric extensions of quadratic form (16)

$$Q_\kappa(A) = \lim_{r \rightarrow 0} \left(\int_{\mathbb{R}^3 \setminus B_r} \left| \frac{\partial A_j}{\partial x_k} \right|^2 d^3x - \left(\frac{5}{3r} + \frac{44}{27}\kappa \right) \int_{\partial B_r} |\vec{A}(\vec{x})|^2 d^2s \right), \quad (19)$$

for transverse vectors $\vec{A}(\vec{x})$ with respect to the scalar product

$$(\vec{A}, \vec{B})_{\mathbb{R}^3} = \int_{\mathbb{R}^3} \overline{A_j(\vec{x})} B_j(\vec{x}) d^3x.$$

By B_r we have denoted a ball of radius r centered at any preferred point. For all real-valued vector fields that are regular at that point (which we shall further take to be the origin) the value of form Q_κ obviously equals the value of form (16)

$$Q_\kappa(A) = Q(A) = \int_{\mathbb{R}^3} \left(\frac{\partial A_j}{\partial x_k} \right)^2 d^3x.$$

But the domain of form Q_κ also includes fields with singularities of type (1) in their three transverse components of angular momentum $l = 1$. The reason for this is that for such fields the singularities of the order r^{-1} in the volume integral in (19) get canceled by the singularities of the integral over

the boundary of ball B_r . Notably, the domains of all non-trivial extensions Q_κ coincide and do not depend on κ . We should also add that the coefficient at the dimensional parameter κ in equation (19) can be taken arbitrary, the value $\frac{44}{27}$ has been chosen to conform to boundary condition (29), which will be introduced later.

Next we note that by the reason that singular fields of the form (1) are inadmissible for higher order terms of Hamiltonian \mathcal{H}_ε we can demand that the basic relations for “eigenfunctionals” $\Phi_{\sigma_n}(A)$ of operator \mathcal{H}_{ren}

$$\mathcal{H}_{\text{ren}} \Phi_{\sigma_n}(A) = \Lambda_{\sigma_n} \Phi_{\sigma_n}(A)$$

be obeyed only on the domain of quadratic form $Q(A)$. But on that domain the above relations would also be obeyed for a quantum operator with form $Q_\kappa(A)$ in place of form $Q(A)$. Form $Q_\kappa(A)$, after taking the square root of and substituting into a Gaussian integral of the type (18), yields a radically different vacuum state and a different set of excitations corresponding to a different operator $\mathcal{H}_{\text{ren}}^\kappa \neq \mathcal{H}_0$. One can propose that operator \mathcal{H}_0 is a self-adjoint extension of some symmetric operator which is defined on a set of functionals rapidly vanishing near boundary functions with singularities of type (1). This symmetric operator also admits other extensions $\mathcal{H}_{\text{ren}}^\kappa$, *i.e.* the ones the “eigenstates” for which are built by means of the quadratic form $Q_\kappa(A)$.

It should be said that any other (possible) extension of form (16) can be used instead of (19) within the present method, we have just picked up the one that we know from [12]. For a more detailed study let us switch to spherical coordinates and single out the subspace of angular momentum $l = 1$ from the field variables.

2.2 Vector spherical harmonics and separation of variables

Using scalar spherical functions $Y_{lm}(\psi, \varphi)$ let us introduce the three vector spherical harmonics (VSH) [13]:

$$\begin{aligned}\vec{Y}_{lm}(\Omega) &= \frac{\vec{x}}{r} Y_{lm}, \quad 0 \leq l, \quad |m| \leq l, \\ \vec{\Psi}_{lm}(\Omega) &= \tilde{l}^{-1} r \vec{\partial} Y_{lm}, \quad 1 \leq l, \quad |m| \leq l, \quad \tilde{l} = \sqrt{l(l+1)}, \\ \vec{\Phi}_{lm}(\Omega) &= \tilde{l}^{-1} (\vec{x} \times \vec{\partial}) Y_{lm}, \quad 1 \leq l, \quad |m| \leq l,\end{aligned}$$

which are functions of angular variables $\Omega = (\psi, \varphi)$. These functions are mutually orthogonal and normalized in terms of integration over the sphere,

$$\begin{aligned} \int_{\mathbb{S}^2} \overline{\Upsilon_{lm}(\Omega)} \vec{\Psi}_{l'm'}(\Omega) d\Omega &= 0, & \int_{\mathbb{S}^2} \overline{\Upsilon_{lm}(\Omega)} \vec{\Upsilon}_{l'm'}(\Omega) d\Omega &= \delta_{ll'} \delta_{mm'}, \\ \int_{\mathbb{S}^2} \overline{\Upsilon_{lm}(\Omega)} \vec{\Phi}_{l'm'}(\Omega) d\Omega &= 0, & \int_{\mathbb{S}^2} \overline{\vec{\Psi}_{lm}(\Omega)} \vec{\Psi}_{l'm'}(\Omega) d\Omega &= \delta_{ll'} \delta_{mm'}, \\ \int_{\mathbb{S}^2} \overline{\vec{\Phi}_{lm}(\Omega)} \vec{\Psi}_{l'm'}(\Omega) d\Omega &= 0, & \int_{\mathbb{S}^2} \overline{\vec{\Phi}_{lm}(\Omega)} \vec{\Phi}_{l'm'}(\Omega) d\Omega &= \delta_{ll'} \delta_{mm'}. \end{aligned}$$

The vector spherical harmonics enable one to uniquely represent a vector function $\vec{A}(\vec{x})$ as three sums

$$\vec{A}(\vec{x}) = \sum_{0 \leq |m| \leq l} y_{lm}(r) \vec{\Upsilon}_{lm} + \sum_{l,m} \chi_{lm}(r) \vec{\Psi}_{lm} + \sum_{l,m} w_{lm}(r) \vec{\Phi}_{lm}. \quad (20)$$

For brevity we shall from now on assume that summation in indices l, m is always taken in the range $1 \leq l, |m| \leq l$ unless stated otherwise explicitly. For each component of expansion (20), when acted upon with Laplacian Δ , the following separation of variable takes place

$$\Delta(z(r) \vec{Z}_{lm}) = -\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} z(r) \vec{Z}_{lm} + \frac{z(r)}{r^2} \Delta_{\Omega} \vec{Z}_{lm}, \quad \vec{Z} = \vec{\Upsilon}, \vec{\Psi}, \vec{\Phi}.$$

The action of the spherical Laplacian Δ_{Ω} on the VSH is non-diagonal (for $l \geq 1$) but with the given normalization it turns out to be symmetric,

$$\begin{aligned} \Delta_{\Omega} \vec{\Upsilon}_{lm} &= (2 + \tilde{l}^2) \vec{\Upsilon}_{lm} - 2\tilde{l} \vec{\Psi}_{lm}, \\ \Delta_{\Omega} \vec{\Psi}_{lm} &= -2\tilde{l} \vec{\Upsilon}_{lm} + \tilde{l}^2 \vec{\Psi}_{lm}, \\ \Delta_{\Omega} \vec{\Phi}_{lm} &= \tilde{l}^2 \vec{\Phi}_{lm}. \end{aligned}$$

If one imposes the condition of transversality (10) upon a vector function $\vec{A}(\vec{x})$, then it will be parametrized by just two sets of functions $u_{lm}(r), w_{lm}(r)$ instead of three as in (20),

$$\vec{A}(\vec{x}) = \sum_{l,m} \left(\tilde{l} \frac{u_{lm}}{r^2} \vec{\Upsilon}_{lm} + \frac{u'_{lm}}{r} \vec{\Psi}_{lm} + \frac{w_{lm}}{r} \vec{\Phi}_{lm} \right). \quad (21)$$

Each of the first two terms in the bracket is not transverse by itself, but the terms become so when taken together,

$$\begin{aligned} \vec{\partial} \cdot \left(\tilde{l} \frac{u_{lm}}{r^2} \vec{\Upsilon}_{lm} + \frac{u'_{lm}}{r} \vec{\Psi}_{lm} \right) = \\ = \tilde{l} Y_{lm} \left(\left(\frac{u'_{lm}}{r^2} - \frac{2u_{lm}}{r^3} \right) \frac{\vec{x}}{r} \cdot \frac{\vec{x}}{r} + \frac{u_{lm}}{r^2} \vec{\partial} \cdot \frac{\vec{x}}{r} \right) + \tilde{l}^{-1} u'_{lm} \vec{\partial} \cdot \vec{\partial} Y_{lm} = 0. \end{aligned} \quad (22)$$

The action of the quadratic form of the Laplace operator on a transverse field $\vec{A}(\vec{x})$ written in terms of new variables $u_{lm}(r)$, $w_{lm}(r)$ takes the following form (see the corresponding equations in [14])

$$Q(A) = \int_{\mathbb{R}^3} A_j(\vec{x}) \Delta A_j(\vec{x}) d^3x = \sum_{l,m} \langle u_{lm}, \tilde{T}_l u_{lm} \rangle_l + \sum_{l,m} (w_{lm}, \tilde{T}_l w_{lm}),$$

where $\langle \cdot, \cdot \rangle_l$ is the scalar product inherited from \mathbb{R}^3

$$\langle u, v \rangle_l = \int_0^\infty \left(\overline{u'(r)} v'(r) + \frac{l(l+1)}{r^2} \overline{u(r)} v(r) \right) dr, \quad u(0) = v(0) = 0, \quad (23)$$

while the radial part of the Laplace operator \tilde{T}_l and the scalar product (\cdot, \cdot) have been defined in (3) and (5). For now we are assuming that functions $u_{lm}(r)$, $w_{lm}(r)$ are smooth enough and fall off rapidly towards the origin. A surprising fact significantly simplifying calculations is that product (23) can be defined as a sesquilinear form of operation T_l in scalar product (\cdot, \cdot)

$$\langle u, v \rangle_l = \int_0^\infty \overline{u(r)} \left(-\frac{d^2}{dr^2} v(r) + \frac{l(l+1)}{r^2} v(r) \right) dr = (u, T_l v). \quad (24)$$

In order to avoid confusion between the differential operation T_l arising from the scalar product and the radial part of the Laplace operator, we have denoted the latter as \tilde{T}_l and will keep this notation later.

The kinetic term of Hamiltonian (15) can be re-written as follows,

$$\begin{aligned} - \iint_{\mathbb{R}^3} \frac{\delta}{\delta A_k(\vec{x})} P_{kj}(\vec{x}, \vec{y}) \frac{\delta}{\delta A_j(\vec{x})} d^3x d^3y = \\ = - \iint \left(\frac{\delta}{\delta w_{l'm'}(r')} \frac{\delta w_{l'm'}(r')}{\delta A_k(\vec{x})} + \frac{\delta}{\delta u_{l'm'}(r')} \frac{\delta u_{l'm'}(r')}{\delta A_k(\vec{x})} \right) P_{kj}(\vec{x}, \vec{y}) \times \\ \times \left(\frac{\delta w_{lm}(r)}{\delta A_j(\vec{y})} \frac{\delta}{\delta w_{lm}(r)} + \frac{\delta u_{lm}(r)}{\delta A_j(\vec{y})} \frac{\delta}{\delta u_{lm}(r)} \right) dr dr' d^3x d^3y. \end{aligned} \quad (25)$$

In order for projector $P_{kj}(\vec{x}, \vec{y})$

$$P(\vec{x}, \vec{y}) = \sum_{l,m} \left(\frac{\tilde{l}}{s} \vec{\Upsilon}_{lm}(\Omega) - \frac{\partial}{\partial s} \vec{\Psi}_{lm}(\Omega) \right) T_l^{-1}(s, r) \left(\frac{\tilde{l}}{r} \overline{\vec{\Upsilon}_{lm}(\Omega')} + \frac{\partial}{\partial r} \overline{\vec{\Psi}_{lm}(\Omega')} \right) + \\ + \sum_{l,m} s^{-1} \vec{\Phi}_{lm}(\Omega) \delta(s-r) \overline{\vec{\Phi}_{lm}(\Omega')} r^{-1}, \quad \vec{x} = (s, \Omega), \quad \vec{y} = (r, \Omega')$$

to act as a unit operator on transverse functions, let us accept the following parametrization for new variables (u_{lm}, w_{lm}) in terms of \vec{A} :

$$w_{lm}(r) = r \int d\Omega \overline{\vec{\Phi}_{lm}(\Omega)} \cdot \vec{A}(r, \Omega) = \frac{1}{r} \int d^3x \delta(r-s) \overline{\vec{\Phi}_{lm}(\Omega)} \cdot \vec{A}(\vec{x}), \quad (26)$$

$$u_{lm}(r) = \int ds T_l^{-1}(r, s) \int d\Omega \left(\tilde{l} \overline{\vec{\Upsilon}_{lm}(\Omega)} - \frac{\partial}{\partial s} s \overline{\vec{\Psi}_{lm}(\Omega)} \right) \cdot \vec{A}(s, \Omega) = \\ = \int d^3x \left(\frac{\tilde{l}}{s^2} T_l^{-1}(r, s) \overline{\vec{\Upsilon}_{lm}(\Omega)} + \frac{1}{s} \left(\frac{\partial}{\partial s} T_l^{-1}(r, s) \right) \overline{\vec{\Psi}_{lm}(\Omega)} \right) \cdot \vec{A}(\vec{x}), \quad (27)$$

where again $\vec{x} = \vec{x}(s, \Omega)$ and T_l^{-1} come from (4). It is not difficult to see that these expressions restore fields $u_{lm}(r)$, $w_{lm}(r)$ from $\vec{A}(\vec{x})$ expressed as (21), while, at the same time, they get annihilated on any longitudinal component

$$\vec{A}^L(\vec{x}) = \sum_{0 \leq |m| \leq l} \left(v'_{lm}(r) \vec{\Upsilon}_{lm}(\Omega) + \frac{\tilde{l}}{r} v_{lm}(r) \vec{\Psi}_{lm}(\Omega) \right) = \vec{\partial} \sum_{0 \leq |m| \leq l} v_{lm}(r) Y_{lm}(\Omega).$$

Let us calculate the variations $\frac{\delta w_{lm}}{\delta A}$, $\frac{\delta u_{lm}}{\delta A}$ from (26), (27) and substitute them into (25). We find,

$$- \int dr dr' d^3x \left(\frac{\delta}{\delta w_{l'm'}(r')} \vec{\Phi}_{l'm'}(\Omega) \frac{\delta(r'-s)}{r'} \cdot \frac{\delta(s-r)}{r} \overline{\vec{\Phi}_{lm}(\Omega)} \frac{\delta}{\delta w_{lm}(r)} + \right. \\ \left. + \frac{\delta}{\delta u_{l'm'}(r')} \left(\frac{\tilde{l}}{s^2} T_l^{-1}(r', s) \vec{\Upsilon}_{l'm'}(\Omega) + \frac{1}{s} \left(\frac{\partial}{\partial s} T_l^{-1}(r', s) \right) \vec{\Psi}_{l'm'}(\Omega) \right) \cdot \right. \\ \left. \cdot \left(\frac{\tilde{l}}{s^2} T_l^{-1}(s, r) \overline{\vec{\Upsilon}_{lm}(\Omega)} + \frac{1}{s} \left(\frac{\partial}{\partial s} T_l^{-1}(s, r) \right) \overline{\vec{\Psi}_{lm}(\Omega)} \right) \frac{\delta}{\delta u_{lm}(r)} \right) = \\ = - \int dr \frac{\delta}{\delta w_{lm}(r)} \frac{\partial}{\partial w_{lm}(r)} - \\ - \int dr' ds dr \frac{\delta}{\delta u_{lm}(r')} T_l^{-1}(r', s) \left(-\frac{\partial^2}{\partial s^2} + \frac{\tilde{l}^2}{s^2} \right) T_l^{-1}(s, r) \frac{\delta}{\delta u_{lm}(r)},$$

where we have immediately dropped the cross terms of u and w which vanish due to orthogonality of the VSH. In the last term the action of T_l^{-1} on T_l produces a δ -function which removes one integration. It is worth noting here that the appearance of coefficient $T_l^{-1}(r, s)$ in the square of conjugate “momenta” (*i.e.* in the kinetic part of the Hamiltonian) is quite natural if the “coordinate” variable $u_l(r)$ is measured by scalar product (24) involving operation T_l .

Adding up the kinetic and potential parts we find the following expression for Hamiltonian (15) in terms of the new variables,

$$\begin{aligned} \mathcal{H}_0 = & \sum_{l,m} \left(- \int_0^\infty dr \frac{\delta}{\delta w_{lm}(r)} \frac{\delta}{\delta w_{lm}(r)} + (w_{lm}, \check{T}_l w_{lm}) \right) + \\ & + \sum_{l,m} \left(- \int \int_0^\infty dr dr' \frac{\delta}{\delta u_{lm}(r')} T_l^{-1}(r', r) \frac{\delta}{\delta u_{lm}(r)} + \langle u_{lm}, \check{T}_l u_{lm} \rangle_l \right). \end{aligned}$$

As was expected, variables w_{lm} , u_{lm} separate for all l and m , while the vacuum states and excitations of this Hamiltonian can be sought as products of states of Hamiltonians

$$\mathcal{H}_{lm} = - \int \int_0^\infty dr dr' \frac{\delta}{\delta u_{lm}(r')} T_l^{-1}(r', r) \frac{\delta}{\delta u_{lm}(r)} + \langle u_{lm}, \check{T}_l u_{lm} \rangle_l$$

and

$$\mathcal{H}'_{lm} = - \int_0^\infty dr \frac{\delta}{\delta w_{lm}(r)} \frac{\delta}{\delta w_{lm}(r)} + (w_{lm}, \check{T}_l w_{lm}).$$

Hamiltonians \mathcal{H}'_{lm} are operators found after switching to spherical coordinates and separating the variables in a Hamiltonian for a free scalar field. For $l \geq 1$ their eigenvectors are evidently defined unambiguously, and so we do not consider them in detail, paying close attention to operators \mathcal{H}_{lm} instead.

2.3 Extensions of quadratic form of operator \check{T}_1

It was shown in [12] that operator \check{T}_1 in scalar product $\langle \cdot, \cdot \rangle_1$ is a symmetric operator with deficiency indices of $(1, 1)$. This operator has non-trivial self-adjoint extensions that act as mixed expressions

$$\check{T}_{1\kappa} u(r) = T_1 u(r) - \frac{2}{r} u'(0) = -\frac{d^2 u(r)}{dr^2} + \frac{2}{r^2} u(r) - \frac{2}{r} u'(0). \quad (28)$$

on the domains

$$\mathcal{D}^\kappa = \{u(r) : \langle u, u \rangle_1 < \infty, \langle \check{T}_{1\kappa} u, \check{T}_{1\kappa} u \rangle_1 < \infty, 3u''(0) = 4u'(0)\}. \quad (29)$$

Operators $\check{T}_{1\kappa}$ have a single-valued continuous spectrum which occupies the non-negative half-axis, and to which the following “eigenfunctions” (the kernel of the spectral transformation) correspond

$$p_{1\lambda}^\kappa(r) = \frac{2r}{\sqrt{2\pi}\lambda^2} \frac{d}{dr} \frac{1}{r} (\cos(\zeta + \lambda r) - \cos \zeta),$$

where the phase shift ζ is defined by

$$e^{2i\zeta} = \frac{\lambda - i\kappa}{\lambda + i\kappa}.$$

At $\kappa < 0$ operator $\check{T}_{1\kappa}$ has an eigenvalue $-\kappa$ of multiplicity one (the discrete spectrum) and an eigenfunction

$$q(r) = q_\kappa(r) = \sqrt{-\frac{2}{\kappa^3}} \left(\kappa e^{\kappa r} + \frac{1 - e^{\kappa r}}{r} \right).$$

The set $\{p_{1\lambda}^\kappa, q\}$ enjoys the conditions of orthogonality

$$\langle p_{1\lambda}^\kappa, p_{1\mu}^\kappa \rangle_1 = \delta(\lambda - \mu), \quad \langle p_{1\lambda}^\kappa, q \rangle_1 = 0, \quad \langle q, q \rangle_1 = 1$$

and completeness,

$$\int_0^\infty p_{1\lambda}^\kappa(r) T_1^s p_{1\lambda}^\kappa(s) d\lambda + q(r) T_1^s q(s) \Big|_{\kappa < 0} = \delta(r - s),$$

where index s of differential operation T_1^s emphasizes that the latter acts on variable s . Operators $\check{T}_{1\kappa}$ generate extensions $\langle u, \check{T}_{1\kappa} u \rangle_1$ of the quadratic forms from the potential parts of Hamiltonians \mathcal{H}_{1m} . The original form $\langle u, \check{T}_1 u \rangle_1$ is defined on the set of doubly differentiable functions vanishing at the origin along with their first derivatives

$$\mathcal{W}_0^2 = \{u(r) : \langle u, u \rangle_1 < \infty, \langle u, \check{T}_1 u \rangle_1 < \infty, u(0) = u'(0) = 0\},$$

which corresponds to differentiable fields

$$\vec{A}(\vec{x}) = \sqrt{2} \frac{u_{1m}(r)}{r^2} \vec{\Upsilon}_{1m}(\psi, \varphi) + \frac{u'_{1m}(r)}{r} \vec{\Psi}_{1m}(\psi, \varphi) \quad (30)$$

regular at the origin. The extended forms $\langle u, \tilde{T}_{1\kappa} u \rangle_1$ are defined on a set of functions with an arbitrary bounded value of the derivative at the origin

$$\mathcal{W}_1^2 = \{u(r) : \langle u, u \rangle_1 < \infty, \langle u, \tilde{T}_{1\kappa} u \rangle_1 < \infty, u(0) = 0\}.$$

Obviously the latter form equals the former on set \mathcal{W}_0^2

$$\langle u, \tilde{T}_{1\kappa} u \rangle_1 = \langle u, \tilde{T}_1 u \rangle_1, \quad u \in \mathcal{W}_0^2$$

as long as the last term in (28) vanishes. We shall not provide a symmetric limiting expression like (19) for the extended form, as we pass right on to the spectral expansion instead,

$$\langle u, \tilde{T}_{1\kappa} u \rangle_1 = \iint_0^\infty Q_\kappa(r, s) T_1^r \overline{u(r)} T_1^s u(s) dr ds,$$

with

$$Q_\kappa(r, s) = \int_0^\infty p_{1\lambda}^\kappa(r) p_{1\lambda}^\kappa(s) \lambda^2 d\lambda - \kappa^2 T_1^r q(r) T_1^s q(s) \Big|_{\kappa < 0},$$

wherein the second term exists only for $\kappa < 0$.

To conclude this subsection we note that form $\langle u, \tilde{T}_1 u \rangle_1$ is in fact a special case of form $\langle u, \tilde{T}_{1\kappa} u \rangle_1$ corresponding to $\kappa = \infty$ (*i.e.* the form of Friedrichs or *maximal* extension of symmetric operator \tilde{T}_1). From the perspective of the spectral properties of these forms, one can observe that the spherical Bessel function

$$p_{1\lambda}(r) = \frac{2r}{\sqrt{2\pi}\lambda^2} \frac{d}{dr} \frac{1}{r} \sin \lambda r,$$

appearing in the parametrization of the non-singular transverse field (30), is a limiting case of function $p_{1\lambda}^\kappa$

$$p_{1\lambda}(r) = \lim_{\kappa \rightarrow \infty} \frac{2r}{\sqrt{2\pi}\lambda^2} \frac{d}{dr} \frac{1}{r} (\cos(\zeta + \lambda r) - \cos \zeta), \quad \zeta(\kappa) \rightarrow -\frac{\pi}{2}.$$

2.4 Hamiltonian eigenstates at $l = 1$

The spectral expansion that we have obtained in the previous subsection now allows us to write the Gaussian functional $\phi_0^\kappa(u)$ for the extended quantum operator

$$\mathcal{H}_{1m}^\kappa = - \iint_0^\infty dr ds \frac{\delta}{\delta u(s)} T_1^{-1}(s, r) \frac{\delta}{\delta u(r)} + \langle u, \tilde{T}_{1\kappa} u \rangle_1$$

as the following exponent of an integral operator

$$\phi_0^\kappa(u) = \exp\left\{-\frac{1}{2} \iint Q_\kappa^{\frac{1}{2}}(r, s) T_1^r u(r) T_1^s u(s) dr ds\right\},$$

where

$$Q_\kappa^{\frac{1}{2}}(r, s) = \int p_{1\lambda}^\kappa(r) p_{1\lambda}^\kappa(s) \lambda d\lambda - i\kappa q(r) q(s) \Big|_{\kappa < 0}.$$

In this expression we have purposely pulled out differential operations T_1^r , T_1^s in order to obtain a more smooth kernel $Q_\kappa^{\frac{1}{2}}$. It is not difficult to see that functional ϕ_0^κ satisfies the equation

$$\mathcal{H}_{1m}^\kappa \phi_0^\kappa(u) = \Lambda_0^\kappa \phi_0^\kappa(u), \quad \Lambda_0^\kappa = \int_0^\infty T_1^r Q_\kappa^{\frac{1}{2}}(r, r') \Big|_{r=r'} dr$$

with some infinite eigenvalue Λ_0^κ . In order to diagonalize operator \mathcal{H}_{1m}^κ let us pass to the spectral representation of the quadratic form, *i.e.* perform the substitution

$$\hat{u}(\lambda) = \int_0^\infty p_{1\lambda}^\kappa(r) T_1 u(r) dr, \quad \hat{u}_d = \int_0^\infty q(r) T_1 u(r) \Big|_{\kappa < 0}$$

(note that all functions here are real), then

$$\mathcal{H}_{1m}^\kappa = \int \left(-\frac{\delta}{\delta \hat{u}(\lambda)} \frac{\delta}{\delta \hat{u}(\lambda)} + \lambda^2 \hat{u}^2(\lambda) \right) d\lambda - \kappa^2 \hat{u}_d^2 \Big|_{\kappa < 0}.$$

This quantum Hamiltonian is associated to the following creation and annihilation operators

$$\hat{b}(\lambda) = \lambda \hat{u}(\lambda) - \frac{\delta}{\delta \hat{u}(\lambda)}, \quad \hat{a}(\lambda) = \lambda \hat{u}(\lambda) + \frac{\delta}{\delta \hat{u}(\lambda)}$$

and to a vacuum state

$$\hat{\phi}_0(\hat{u}) = \phi_0(u(\hat{u})) = \exp\left\{-\frac{1}{2} \int_0^\infty \hat{u}^2(\lambda) \lambda d\lambda + \frac{i\kappa}{2} \hat{u}_d^2 \Big|_{\kappa < 0}\right\}.$$

n -particle eigenstates are constructed as integrals with Bose-Einstein coefficients $\sigma(\lambda_1, \dots, \sigma_{\lambda_n})$

$$\hat{\phi}_{\sigma_n}(\hat{u}) = \iint \sigma_n(\lambda_1, \dots, \lambda_n) \hat{b}(\lambda_1) \dots \hat{b}(\lambda_n) d\lambda_1 \dots d\lambda_n \hat{\phi}_0(\hat{u}), \quad (31)$$

and, furthermore, for $\kappa < 0$ there are states related to the excitations of the discrete spectrum.

2.5 Eigenstates of the quantum Hamiltonian of a free transverse field

The eigenstates of quantum Hamiltonian $\mathcal{H}_{\text{ren}}^\kappa$, involving the extended quadratic form (19), are constructed as products of eigenstates of operators \mathcal{H}'_{lm} , with $1 \leq l, |m| \leq l$, \mathcal{H}_{lm} with $2 \leq l, |m| \leq l$ and \mathcal{H}_{1m}^κ . For diagonalization of the first two sets of operators one can exploit the standard spectral transformation

$$\hat{u}_{lm}(\lambda) = \int_0^\infty p_{l\lambda}(r) T_l u_{lm}(r) dr, \quad \hat{w}_{lm}(\lambda) = \int_0^\infty \lambda p_{l\lambda}(r) w_{lm}(r) dr,$$

where $p_{l\lambda}(r)$ are some kind of spherical Bessel functions

$$p_{l\lambda}(r) = \frac{2r^l}{\sqrt{2\pi}\lambda^{l+1}} \left(\frac{d}{dr} \frac{1}{r} \right)^l \sin \lambda r.$$

The corresponding creation and annihilation operators as well as the vacuum states look as follows,

$$\begin{aligned} \hat{b}_{lm}(\lambda) &= \lambda \hat{u}_{lm}(\lambda) - \frac{\delta}{\delta \hat{u}_{lm}(\lambda)}, & \hat{a}_{lm}(\lambda) &= \lambda \hat{u}_{lm}(\lambda) + \frac{\delta}{\delta \hat{u}_{lm}(\lambda)} \\ \hat{b}'_{lm}(\lambda) &= \lambda \hat{w}_{lm}(\lambda) - \frac{\delta}{\delta \hat{w}_{lm}(\lambda)}, & \hat{a}'_{lm}(\lambda) &= \lambda \hat{w}_{lm}(\lambda) + \frac{\delta}{\delta \hat{w}_{lm}(\lambda)}, \\ \hat{\phi}_0(\hat{u}_{lm}) &= \exp\left\{-\frac{1}{2} \int_0^\infty \hat{u}_{lm}^2(\lambda) \lambda d\lambda\right\}, \\ \hat{\phi}'_0(\hat{w}_{lm}) &= \exp\left\{-\frac{1}{2} \int_0^\infty \hat{w}_{lm}^2(\lambda) \lambda d\lambda\right\}. \end{aligned}$$

Diagonalization of Hamiltonian \mathcal{H}_{1m}^κ via the transformation

$$\hat{u}_{1m}(\lambda) = \int_0^\infty p_{1\lambda}^\kappa(r) T_1 u_{1m}(r) dr,$$

has been described in the previous subsection. It is worth to note here that in a spherically non-symmetric case coefficients κ may be different for components corresponding to different values m of the third component of the angular momentum.

In terms of variables \hat{u}_{lm} , \hat{w}_{lm} we find the resulting Hamiltonian

$$\hat{\mathcal{H}}_{\text{ren}}^\kappa = \sum_{-1 \leq m \leq 1} \hat{\mathcal{H}}_{1m}^\kappa + \sum_{2 \leq l, |m| \leq l} \hat{\mathcal{H}}_{lm} + \sum_{1 \leq l, |m| \leq l} \hat{\mathcal{H}}'_{lm},$$

and the vacuum state

$$\Phi_0^\kappa = \prod_{-1 \leq m \leq 1} \phi_{1m}(\hat{u}_{1m}) \times \prod_{2 \leq l, |m| \leq l} \phi_{lm}(\hat{u}_{lm}) \times \prod_{l,m} \phi'_{lm}(\hat{w}_{lm}),$$

while the n -particle states are obtained from Eq. (31) by replacing creation operator $\hat{b}(\lambda)$ with an operator $c(\lambda)$ and $\hat{\phi}_0$ with $\hat{\Phi}_0^\kappa$

$$\hat{\Phi}_{\sigma_n}(\{\hat{u}\}) = \iint \sigma_n(\lambda_1, \dots, \lambda_n) c(\lambda_1) \dots c(\lambda_n) d\lambda_1 \dots d\lambda_n \hat{\Phi}_0(\{\hat{u}\}),$$

where $c(\lambda)$ can take either of the values $\hat{b}_{lm}(\lambda)$ or $\hat{b}'_{lm}(\lambda)$.

3 Conclusion and discussion

We have constructed a system of states satisfying the eigenstate equations for a quantum Hamiltonian operator of a free transverse field. The resulting sets generally depend on a preferred point in space and do not possess scale invariance (*i.e.* depend on a dimensional parameter). Our construction has heavily relied upon the properties of extensions of the quadratic form of the Laplace operator which appears in the potential term of the Hamiltonian. These extensions can be written in an invariant form (19) which, similar to the transversality condition, does not imply a transition to spherical coordinates or using any preferred functional parametrization of the type of (21). A natural question arises here about a possibility to generalize form (19) to the case of two or more preferred points in the space

$$Q_{\{\kappa\}}(A) = \lim_{r \rightarrow 0} \left(\int_{\mathbb{R}^3 \setminus \{B_{r,n}\}} \left(\frac{\partial A_k}{\partial x_j} \right)^2 d^3x - \sum_{n=1}^N \left(\frac{5}{3r} + \kappa_n \right) \int_{\partial B_{r,n}} |\vec{A}(\vec{x})|^2 d^2s \right),$$

— would such a form satisfy the conditions of theorem VIII.15 of [11], does it have the corresponding self-adjoint operator, and if it does, can a spectral representation be found for the latter? A significant difficulty here can be in the transversality condition. It is not clear, whether it survives the closure *w. r.* to the norm provided by the scalar product and the quadratic form. For the case of a single preferred point there already exists a spectral representation for the transverse functions, which further enables us to discuss related physics, while for the case of several preferred points there may be no such representation at all.

Another important remark is that, seemingly, representation of a physical object (a field mediating an interaction) in terms of a vector function on a three-dimensional space is not the right method to describe the problem. While two functions with singularities at two different points may represent a single physical object at different moments in time, they cannot be expressed in terms of a common basis, *i.e.* they do not have a common representation in terms of a single orthogonal set. This, therefore, creates a significant obstacle for describing the possible dynamics of the system.

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